Assignment 10

Hand in no. 2, 4, 5, and 8 by Nov 28, 2023.

1. Let E be a bounded, convex set in $\mathbb{R}^n$. Show that a family of equicontinuous functions is bounded in E if it is bounded at a single point, that is, if there are $x_0 \in E$ and constant M such that $|f(x_0)| \leq M$ for all f in this family.

2. Let $\{f_n\}$ be a sequence of bounded functions in $[0,1]$ and let F_n be

$$F_n(x) = \int_0^x f_n(t)dt.$$

   (a) Show that the sequence $\{F_n\}$ has a convergent subsequence provided there is some M such that $\|f_n\|_\infty \leq M$, for all n.

   (b) Show that the conclusion in (a) holds when boundedness is replaced by the weaker condition: There is some K such that

$$\int_0^1 |f_n|^2 \leq K, \quad \forall n.$$

3. Prove that the set consisting of all functions G of the form

$$G(x) = \sin^2 x + \int_0^x \frac{g(y)}{1+g^2(y)}\, dy \ ,$$

   where $g \in C[0,1]$ is precompact in $C[0,1]$.

4. Let $K \in C([a,b] \times [a,b])$ and define the operator T by

$$(Tf)(x) = \int_a^b K(x,y)f(y)dy.$$

   (a) Show that T maps $C[a,b]$ to itself.

   (b) Show that whenever $\{f_n\}$ is a bounded sequence in $C[a,b]$, $\{Tf_n\}$ contains a convergent subsequence.

5. Let f be a bounded, uniformly continuous function on $\mathbb{R}$. Let $f_a(x) = f(x-a)$. Show that there exists a sequence of unit intervals $I_k = [n_k, n_k+1]$, $n_k \to \infty$, such that $\{f_{n_k}\}$ converges uniformly on $[0,1]$.

6. Optional. A bump function is a smooth function φ in $\mathbb{R}^2$ which is positive in the unit disk, vanishing outside the ball, and satisfies $\iint_{\mathbb{R}^2} \varphi(x)\, dA(x) = 1$. Let f be a continuous function defined in an open set containing $\overline{G}$ where G is bounded and open in $\mathbb{R}^2$. For small $\varepsilon > 0$, define

$$f_\varepsilon(x) = \frac{1}{\varepsilon^2} \iint_{\mathbb{R}^2} \varphi\left(\frac{y-x}{\varepsilon}\right) f(y)\, dA(y) \ .$$

   Show that f_ε is $C^\infty(\overline{G})$ and tends to f uniformly as $\varepsilon \to 0$.

   Note. This property has been used in the proof of Cauchy-Peano theorem.

7. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $\mathbb{R}$ (you may draw a table):

   (a) $A = \{n/2^m : \ n, m \in \mathbb{Z}\}$,

   (b) B, all irrational numbers,

   (c) $C = \{0, 1, 1/2, 1/3, \cdots\}$,

   (d) $D = \{1, 1/2, 1/3, \cdots\}$,

   (e) $E = \{x : \ x^2 + 3x - 6 = 0 \}$,

   (f) $F = \cup_k (k, k+1), k \in \mathbb{N}$,

8. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $C[0, 1]$ (you may draw a table):

   (a) $\mathcal{A}$, all polynomials whose coefficients are rational numbers,

   (b) $\mathcal{B}$, all polynomials,

   (c) $\mathcal{C} = \{f : \ \int_0^1 f(x)dx \neq 0\}$,

   (d) $\mathcal{D} = \{f : \ f(1/2) = 1 \}$.